CHAPTER 1

TOOLS

1.1 Fourier Transforms

The *Fourier transform* (FT) is a mathematical operation that yields the spectral content of a signal (Bracewell 1978). It is named after the French mathematician Jean Baptiste Joseph Fourier (1768–1830). If a signal consists of oscillation at a single frequency (e.g., 163 Hz), then its FT will contain a peak at that frequency (Figure 1.1a). If the signal contains a superposition of tones at multiple frequencies, the FT operation essentially provides a histogram of that spectral content (Figure 1.1b). For example, consider the following physical analogy. Suppose several keys on a piano are struck simultaneously and the resultant sounds are sampled and digitized. The FT of that signal will provide information about which keys were struck and with what force.

Fourier transforms are ubiquitous in the practical reconstruction of MR data and also in the theoretical analysis of MR processes. This is because the physical evolution of the transverse magnetization is described very naturally by the FT. In Magnetic Resonance Imaging (MRI), we usually use *complex* Fourier transforms, which employ the complex exponential, rather than separate sine or cosine Fourier transforms. This choice is made because a complex exponential conveniently represents the precession of the magnetization vector. Table 1.1 reviews some basic properties of the complex exponential. Often a *magnitude* operation (i.e., $|Z|$) is used on a pixel-by-pixel basis to convert the complex output of the FT to positive real numbers that can be more conveniently displayed as pixel intensities.

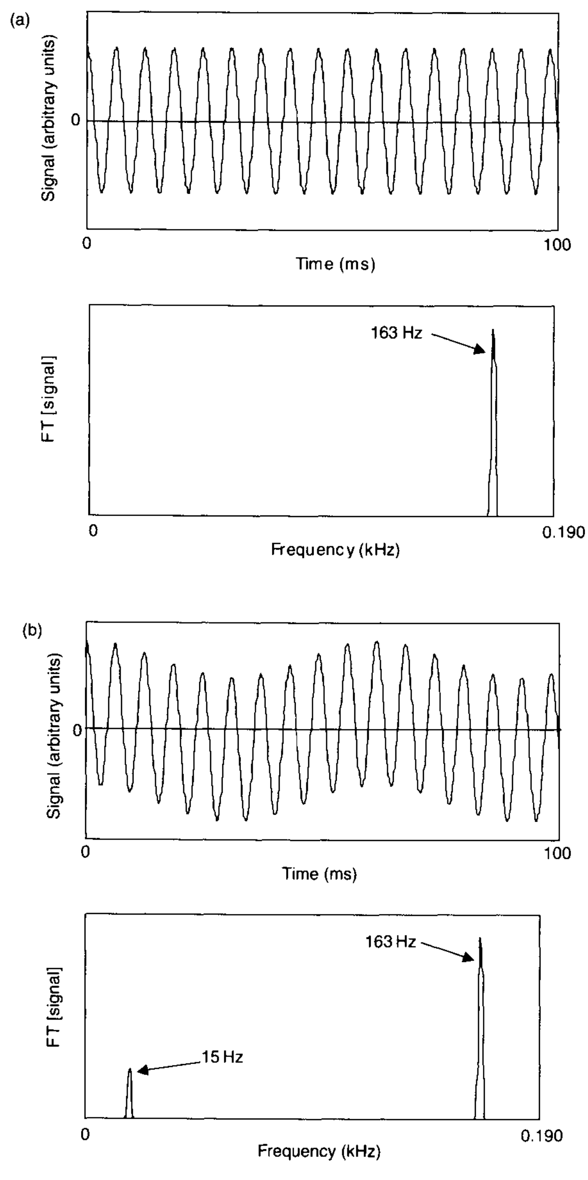

FIGURE 1.1 Schematic representations of the Fourier transform. (a) If a time domain signal contains a tone at a single frequency, its Fourier transform will contain a peak at that frequency, which in this case is 163 Hz. (b) If the signal contains a superposition of two tones, the Fourier transform displays a second peak. In the case shown, a 15-Hz tone with approximately one-quarter the amplitude is modulating the original tone.

TABLE 1.1
Properties of Complex Numbers

$i = \sqrt{-1} = e^{i\pi/2}$
$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{i(a+b)} = e^{ia}\,e^{ib}$
$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$
$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$
if $Z = \rho\, e^{i\theta}$, then $\lvert Z\rvert = \rho$, $\arg(Z) = \measuredangle Z = \theta$
if $Z = x + iy$, then $\lvert Z\rvert = \sqrt{x^2 + y^2}$, $\arg(Z) = \tan^{-1}\left(\frac{y}{x}\right)$
$Z^* = x - iy = \rho e^{-i\theta}$ (complex conjugate of Z)
$Z_1 Z_2 = \rho_1\rho_2 e^{i(\theta_1+\theta_2)} = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

When we are provided with a function of a continuous variable, its FT is calculated by a process that includes integration. This continuous FT is widely used for theoretical work in MRI. The actual MRI signal that is measured, however, is sampled at a finite number of discrete time points, so instead a *discrete Fourier transform* (DFT) is used for practical image reconstruction. With the DFT, the integration operation of the FT is replaced by a finite summation. An important special case of the DFT is called the *fast Fourier transform* (FFT) (Cooley and Tukey 1965; Brigham 1988). The FFT is an algorithm that calculates the DFT of signals whose lengths are particular values (most typically equal to a power of 2, e.g., $256 = 2^8$). As its name implies, the FFT is computationally faster than the standard DFT.

1.1.1 THE CONTINUOUS FOURIER TRANSFORM AND ITS INVERSE

Let $g(x)$ be a function of the real variable x. The output of the function $g(x)$ can have complex values. The complex Fourier transform of $g(x)$ is another function, which we call $G(k)$:

$$\mathrm{FT}\,[g(x)] = G(k) = \int_{-\infty}^{+\infty} g(x)e^{-2\pi ikx}dx \qquad (1.1)$$

The two real variables x and k are known as Fourier *conjugates* and represent a pair of FT *domains*. Examples of domain pairs commonly used in MR are (time, frequency) and (distance, k-space). If the physical units of the pair of variables that represent the two domains are multiplied together, the result is always dimensionless. For example, with the time–frequency pair, the product:

$$1 \text{ millisecond} \times 1\,\text{kHz} = 1 \text{ (dimensionless)} \tag{1.2}$$

The two functions $g(x)$ and $G(k)$ in Eq. (1.1) are called *Fourier transform pairs*. Knowledge about one of the pair is sufficient to reconstruct the other. If $G(k)$ is known, then $g(x)$ can be recovered by performing an *inverse Fourier transform* (IFT):

$$\mathrm{FT}^{-1}[G(k)] = g(x) = \int_{-\infty}^{+\infty} G(k)\, e^{+2\pi i k x} dk \tag{1.3}$$

The IFT undoes the effect of the FT, that is:

$$\mathrm{FT}^{-1}\big[\mathrm{FT}\big[g(x)\big]\big] = g(x) \tag{1.4}$$

and vice versa:

$$\mathrm{FT}\big[\mathrm{FT}^{-1}\big[G(k)\big]\big] = G(k) \tag{1.5}$$

Note that the right sides of Eqs. (1.4) and (1.5) are simply $g(x)$ and $G(k)$, respectively, and are not multiplied by any scaling factors. This is because the IFT definition in Eq. (1.3) is properly *normalized.* A further discussion of the normalization is given in subsection 1.1.10.

Note the factor of 2π that appears in the argument of the exponentials in Eqs. (1.1) and (1.3). If instead domain variables such as time and *angular* frequency (ω, measured in radians/second) are used, then the form of the FT appears somewhat differently. The FT and its inverse become:

$$\begin{aligned} G(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\omega)\, e^{-i\omega t} d\omega \\ g(\omega) &= \int_{-\infty}^{+\infty} G(t)\, e^{i\omega t} dt \end{aligned} \tag{1.6}$$

Note the absence of the 2π factor in the exponential in Eq. (1.6) and the extra multiplicative normalization factor in front of the FT. Equation (1.6) could be recast into a more symmetric form by splitting the 2π into equal $\sqrt{(2\pi)}$ factors in the denominators of both the FT and IFT definitions. Alternatively, we can recast Eq. (1.6) by making the familiar substitution from angular frequency ω to standard frequency f (measured in cycles/second or hertz):

$$\omega = 2\pi f \qquad d\omega = 2\pi\, df \tag{1.7}$$

Substituting Eq. (1.7) into Eq. (1.6) yields the symmetric FT pairs

$$G(t) = \int_{-\infty}^{+\infty} g(f)\, e^{-2\pi i f t} df \tag{1.8}$$

and

$$g(f) = \int_{-\infty}^{+\infty} G(t)\, e^{2\pi i f t} dt \tag{1.9}$$

In this book, we mainly use the form of the FT and IFT with the factor of 2π in the exponential, such as Eqs. (1.1) and (1.8).

1.1.2 MULTIDIMENSIONAL FOURIER TRANSFORMS, AND SEPARABILITY

Multidimensional FTs often arise in MRI. For example, the *two-dimensional FT* (2D-FT) of a function of two variables can be defined as:

$$\begin{aligned} \mathrm{FT}[g(x, y)] = G(k_x, k_y) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x, y)\, e^{-2\pi i k_x x} e^{-2\pi i k_y y} dx\, dy \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x, y)\, e^{-2\pi i \vec{k}\cdot\vec{r}} dx\, dy \end{aligned} \tag{1.10}$$

where $\vec{r} = (x,\ y)$ and $\vec{k} = (k_x, k_y)$ are vectors. The inverse 2D-FT is given by:

$$\mathrm{FT}^{-1}[G(k_x, k_y)] = g(x, y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(k_x, k_y)\, e^{+2\pi i \vec{k}\cdot\vec{r}} dk_x\, dk_y \tag{1.11}$$

Eqs. (1.10) and (1.11) are readily generalized to three or more dimensions.

If the function g is *separable* in x and y:

$$g(x, y) = g_x(x)\, g_y(y) \tag{1.12}$$

then the FT is also separable:

$$\mathrm{FT}[g(x, y)] = \mathrm{FT}[g_x(x)\, g_y(y)] = G_x(k_x)\, G_y(k_y) \tag{1.13}$$

An example of a separable two-dimensional function is the Gaussian:

$$e^{\frac{-(x^2+y^2)}{2\sigma^2}} = e^{\frac{-x^2}{2\sigma^2}}\, e^{\frac{-y^2}{2\sigma^2}} \tag{1.14}$$

In contrast,

$$(x + y)^2 = x^2 + y^2 + 2xy \tag{1.15}$$

is not separable.

1.1.3 Properties of the Fourier Transform

An important property of the FT is the *shift theorem*. A shift or offset of the coordinate in one domain results in a multiplication of the signal by a *linear phase ramp* in the other domain, and vice versa:

$$\mathrm{FT}[g(x + a)] = \int_{-\infty}^{+\infty} g(x)\, e^{-2\pi i k(x+a)}\, dx = G(k)\, e^{-2\pi i k a} \tag{1.16}$$

A second useful property of the FT is that *convolution* in one domain is equivalent to simple *multiplication* in the other. If $f(x)$ and $g(x)$ are two functions, then convolution is defined as:

$$f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(x - x')\, g(x')\, dx' \tag{1.17}$$

and

$$\mathrm{FT}[f(x) \otimes g(x)] = F(k)\, G(k) \tag{1.18}$$

Parseval's theorem (named after Marc-Antoine Parseval des Chêsnes, 1755–1836, a French mathematician) is a third commonly used property of the FT. It states that if f and g are two functions with Fourier transforms F and G, respectively, then

$$\int_{-\infty}^{+\infty} f^*(x)\, g(x)\, dx = \int_{-\infty}^{+\infty} F^*(k)\, G(k)\, dk \tag{1.19}$$

where * denotes complex conjugation. Letting $g = f$ in Eq. (1.19) results in a useful special case, which shows that the FT operation conserves normalization:

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |F(k)|^2\, dk \tag{1.20}$$

Table 1.2 provides several 1D-FT pairs that are commonly used in MRI. These relationships can be applied to multidimensional FTs if the variables are separable.

1.1.4 THE DISCRETE FOURIER TRANSFORM AND ITS INVERSE

In MRI, the sampling process provides a finite number (e.g., 256) of complex data points, rather than a function of a continuous variable. Consequently, the MR image is normally reconstructed with a DFT. Given a string of N complex data points:

$$\{d\} = \{d_0, d_1, d_2, \ldots, d_{N-1}\} \tag{1.21}$$

the Jth element of DFT is defined as:

$$\mathrm{DFT}[\{d\}]_J = D_J = \sum_{K=0}^{N-1} d_K e^{\frac{-2\pi i J K}{N}}, \quad J = 0, 1, 2, \ldots, N-1 \tag{1.22}$$

Note that the index $J = 0$ represents the DC, or zero-frequency element, of the DFT (DC is adopted from the abbreviation for direct current used in electrical engineering). The exponential factor in Eq. (1.22) is sometimes called a *twiddle factor*. The Kth element of the *inverse DFT* (IDFT) is defined as

$$\mathrm{DFT}^{-1}[\{D\}]_K = d_K = \frac{1}{N} \sum_{J'=0}^{N-1} D_{J'} e^{\frac{+2\pi i J' K}{N}}, \quad K = 0, 1, 2, \ldots, N-1 \tag{1.23}$$

The factor of $1/N$ in Eq. (1.23) is required for normalization, so that

$$\begin{aligned} \mathrm{DFT}^{-1}\big[\,\mathrm{DFT}[\{d\}]\big] &= \{d\} \\ \mathrm{DFT}\big[\,\mathrm{DFT}^{-1}[\{D\}]\big] &= \{D\} \end{aligned} \tag{1.24}$$

In analogy to the manipulation of the 2π-normalization factor of the complex FT described for Eqs. (1.6)–(1.9), the normalization factor of $1/N$ in Eq. (1.23) can be moved from the IDFT to the DFT. Alternatively, it is sometimes symmetrically distributed as equal $1/\sqrt{N}$ factors on both the DFT and IDFT. It is

TABLE 1.2
Fourier Transform Pairs Commonly Used in Magnetic Resonance Imaging

$g(x)$	$\mathrm{FT}[g(x)] = G(k) = \int_{-\infty}^{+\infty} g(x)e^{-2\pi ikx}\,dx$
$g(x)e^{2\pi ik_0x}$	$G(k-k_0)$
$g(x-x_0)$	$G(k)e^{-2\pi ikx_0}$
$g\left(\frac{x}{\alpha}\right)$	$\lvert\alpha\rvert G(\alpha k)$
$g(-x)$	$G(-k)$
$\frac{dg(x)}{dx}$	$2\pi ikG(k)$
$g^*(x)$	$G^*(-k)$
$xg(x)$	$\frac{i}{2\pi}\frac{dG(k)}{dk}$
$af(x)+bg(x)$	$aF(k)+bG(k)$
$f(x)\otimes g(x)$	$F(k)G(k)$
$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \int_{-\infty}^{+\infty} \delta(x)\,dx = 1 & \end{cases}$	1
1	$\delta(k)$
$e^{2\pi ik_0x}$	$\delta(k-k_0)$
$\cos(2\pi k_0x)$	$\frac{1}{2}[\delta(k-k_0)+\delta(k+k_0)]$
$\sin(2\pi k_0x)$	$\frac{1}{2i}[\delta(k-k_0)-\delta(k+k_0)]$
$\cos^2(2\pi k_0x)$	$\frac{1}{2}\left[\delta(k)+\frac{\delta(k-2k_0)+\delta(k+2k_0)}{2}\right]$
$\sin^2(2\pi k_0x)$	$\frac{1}{2}\left[\delta(k)-\frac{\delta(k-2k_0)+\delta(k+2k_0)}{2}\right]$
$\cos^3(2\pi k_0x)$	$\frac{1}{8}[3(\delta(k-k_0)+\delta(k+k_0))+\delta(k-3k_0)+\delta(k+3k_0)]$
$\sin^3(2\pi k_0x)$	$\frac{1}{8i}[3(\delta(k-k_0)-\delta(k+k_0))-\delta(k-3k_0)+\delta(k+3k_0)]$

TABLE 1.2
Continued

$\sum_{n=-\infty}^{\infty} \delta(x - nx_0)$	$\frac{1}{x_0} \sum_{m=-\infty}^{\infty} \delta\left(k - \frac{m}{x_0}\right)$
$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$	$\frac{1}{2\pi}\left[\pi\delta(k) - i \text{ Principle part}\left(\frac{1}{k}\right)\right]$
$\text{RECT}\left(\frac{x}{x_0}\right) = \begin{cases} 1 & \text{if } \lvert x\rvert \le x_0 \\ 0 & \text{if } \lvert x\rvert > x_0 \end{cases}$	$2x_0\text{SINC}(u), \quad \text{SINC}(u) \equiv \frac{\sin(u)}{u}, \quad u = 2\pi k x_0$
$e^{-x^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-2\pi^2\sigma^2k^2}, \quad \sigma$ real
$e^{-\lambda\lvert x\rvert}$	$\frac{2\lambda}{\lambda^2 + (2\pi k)^2}, \quad \text{Re}\,\lambda > 0$
$H(x)e^{-\lambda x}$	$\frac{1}{\lambda + 2\pi i k}, \quad \text{Re}\,\lambda > 0$

best to check the documentation of the particular numerical routines that you use. The $1/N$ normalization factor is required (somewhere) for Eq. (1.24) to hold, because:

$$\sum_{K=0}^{N-1} e^{\frac{-2\pi i K(J'-J)}{N}} = \begin{cases} N, & J = J' \\ 0, & \text{otherwise} \end{cases} \tag{1.25}$$

Although the DFT is typically evaluated numerically, it does have some useful analytical properties, which are summarized in Table 1.3.

1.1.5 IDENTIFYING PHYSICAL UNITS WITH THE DISCRETE FOURIER TRANSFORM OUTPUT

In the complex Fourier transform of Eq. (1.1), it is easy to identify physical units with x (e.g., cm) and k (e.g., cm^{-1}). With the DFT of Eq. (1.22), however, J and K are simply dimensionless integer indices. When we perform a DFT on a signal that has physical meaning, such as MRI data, how do we associate physical units with the string of numbers that is the output of the computer? Consider a one-dimensional MR signal $S(k)$, which can be reconstructed with an IFT:

$$I(x) = \int_{-\infty}^{+\infty} S(k)e^{+2\pi i k x}dk \tag{1.26}$$

TABLE 1.3
Discrete Fourier Transform Pairs

d_K	$\mathrm{DFT}[\{d\}]_J = D_J = \sum_{K=0}^{N-1} d_K e^{\frac{-2\pi i JK}{N}}$
$ad_K + bc_K$	$aD_J + bC_J$
$d_K e^{\frac{-2\pi i Ka}{N}}$	D_{J+a}
d_{K-a}	$D_J e^{-\frac{2\pi i Ja}{N}}$
$(-1)^K d_K$	$D_{J+\frac{N}{2}}$
1	$\frac{1-e^{-2\pi i J}}{1-e^{\frac{-2\pi i J}{N}}}$
$d_{K\pm N}$	D_J
d_{N-1-K}	$D_{-J} e^{\frac{2\pi i J}{N}}$

In order to see how Eq. (1.26) relates to the DFT, first approximate the continuous variables x and k with their discrete representations:

$$\begin{aligned} x_J &= J\Delta x \\ k_M &= M\Delta k \end{aligned} \tag{1.27}$$

Approximating the integral by a discrete summation and substituting Eq. (1.27) into Eq. (1.26) yields:

$$I(J\Delta x) = C \sum_{M=-\infty}^{\infty} S(k_M) e^{+2\pi i JM(\Delta k \Delta x)} \tag{1.28}$$

where the constant C is determined by the normalization conditions; that is, it is equivalent to a factor of Δk that converts the integral to a sum. The important point is, comparing the exponentials in Eqs. (1.23) and (1.28), we conclude that:

$$\Delta x \Delta k = \frac{1}{N} \tag{1.29}$$

Equation (1.29) provides the link between the step sizes of the input and output of the DFT operation and the number of complex points in the data string. It is a very useful relationship for MRI. For example, it tells us that the product of the pixel size and the step size in k-space is equal to the inverse of the number of sampled points. Equation (1.29) can be rearranged as:

$$N\Delta k = \frac{1}{\Delta x} \tag{1.30}$$

which says that the total extent in k-space is equal to the inverse of the spatial pixel size. Similarly, the field of view is the inverse of the step size in k-space.

Example 1.1 Suppose 256 complex points are sampled for a total duration of 8.192 ms. An image is reconstructed with a 256-point DFT. Find the bandwidth, and express it in two common forms: bandwidth per pixel $\Delta\nu_{pp}$ and the half-bandwidth $\pm\Delta\nu$.

Answer Applying Eq. (1.30) with the time–frequency domain pairs, the total bandwidth is:

$$N\Delta\nu_{pp} = 2\Delta\nu = \frac{1}{\Delta t} = \frac{256}{8.192 \text{ ms}} = 31.25 \text{ kHz} \tag{1.31}$$

Thus $\Delta\nu = \pm 15.625$ kHz. The bandwidth in units of hertz per pixel is:

$$\Delta\nu_{pp} = \frac{1}{N\Delta t} = \frac{1}{8.192 \text{ ms}} = 122 \text{ Hz/pixel}$$

1.1.6 Properties of the Discrete Fourier Transform

Like the continuous FT, the DFT obeys a shift theorem as well. Similar to Eq. (1.16), multiplying the data by a linearly increasing phase ramp results in the shift:

$$D_{J+a} = \sum_{K=0}^{N-1} d_K e^{\frac{-2\pi i (J+a) K}{N}} = \sum_{K=0}^{N-1} \left(d_K e^{\frac{-2\pi i a K}{N}} \right) e^{\frac{-2\pi i J K}{N}} \tag{1.32}$$

A common case is called the *half field of view* or *Nyquist shift* (named after Harry Nyquist, 1889–1976, a Swedish-born American engineer). Setting $a = N/2$, Eq. (1.32) becomes:

$$D_{J+\frac{N}{2}} = \sum_{K=0}^{N-1} \left(d_K e^{-\pi i K} \right) e^{\frac{-2\pi i J K}{N}} = \sum_{K=0}^{N-1} \left[d_K (-1)^K \right] e^{\frac{-2\pi i J K}{N}} \tag{1.33}$$

The Nyquist shift is used when we want the DC component of the DFT to be centered, instead of occurring at the zeroth point. Because this is typically the case in MR images, the two-dimensional raw data are multiplied by a

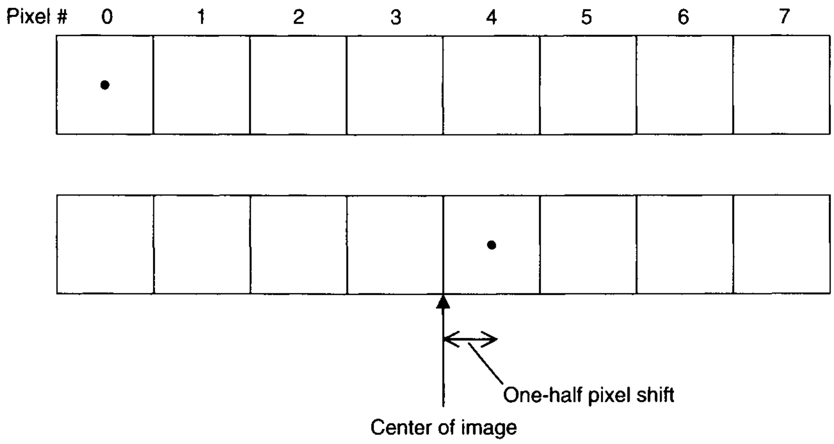

FIGURE 1.2 Demonstration of the one-half pixel shift that arises with the discrete FT. Here the DFT length is $N = 8$. The zero frequency, or DC, component of the DFT occupies the zeroth pixel, whose center is indicated with a dot. After a Nyquist shift, the DC signal occupies the fourth pixel. The center of the fourth pixel is shifted by one-half pixel compared to the image center.

checkerboard $[1, -1]$ pattern before reconstruction (see Section 13.1). Note that if N is an even integer and the $(-1)^K$ multiplier is used to center the DFT response, then the DC component will not be exactly centered but, rather, offset by one-half pixel. Figure 1.2 illustrates using an example with $N = 8$. The shift constant a in Eq. (1.32) need not necessarily be an integer, so shifts by fractions of a pixel can be accomplished with the appropriate linear phase ramp. This is sometimes used to correct the one-half pixel offset.

1.1.7 MULTIDIMENSIONAL DISCRETE FOURIER TRANSFORMS

The two-dimensional DFT can be defined as:

$$D_{JH} = \sum_{L=0}^{M-1} \sum_{K=0}^{N-1} d_{KL} e^{\frac{-2\pi i JK}{N}} e^{\frac{-2\pi i LH}{M}}, \quad J = 0, 1, 2, \ldots, N-1, \quad H = 0, 1, 2, \ldots, M-1 \tag{1.34}$$

Three-dimensional (and higher) DFTs are defined analogously to Eq. (1.34). Note that the data input for the 2D-DFT is a rectilinear $N \times M$ matrix of complex numbers, d_{KL}. Because this matrix can be rearranged either as series of rows or columns, 2D-DFTs can be evaluated by performing either the row or column 1D-DFT first. Similarly, 3D (or higher dimensionality) DFTs can be evaluated by performing the 1D-DFTs in any order. This is a very useful property for partial Fourier reconstructions (Section 13.4), where it is necessary to perform the reconstruction along the partial Fourier direction last.

1.1.8 DISCRETENESS AND PERIODICITY

If the signal in one domain consists of discretely sampled points, then the FT (or IFT) in the other domain is *periodic*. In MRI, the signal is discretely sampled, so the image is periodic or consists of *replicates* that can lead to aliasing artifacts. (This is discussed in more detail in Section 11.1.)

To understand how this property of periodicity arises, consider the continuous signal $S(t)$, $-\infty < t < \infty$. In practice, the signal will be sampled over a finite time interval, $-T < t < T$. To represent the finite sampling interval, first we multiply the signal $S(t)$ by a rectangle (RECT) function, which is defined in Table 1.2 and which is zero for $|t| > T$. Next, the discreteness of the sampling process can be mathematically represented by multiplication by a sampling comb, which consists of a series of Dirac deltas (further described in Section 1.1.10). If the samples are separated in time by Δt, then the sampled signal becomes:

$$S'(t) = S(t)\mathrm{RECT}\left(\frac{t}{T}\right) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \qquad (1.35)$$

According to the convolution theorem, the FT of product in Eq. (1.35) is the convolution of the FTs of the three factors. From Table 1.2, the FT of the RECT function is a sine x over x, or SINC function. Convolution with this factor gives rise to the shape of the point-spread function. The FT of the sampling comb is another series of deltas, this time spaced by $1/\Delta t$. Convolution with this comb gives rise to the periodic nature of the image. In order to avoid aliasing, the replicates must not overlap. This is accomplished by satisfying the *Nyquist criterion*—that is, the sampling rate $1/\Delta t$ must be at least twice the highest frequency contained in the signal. (These properties are further discussed in Section 11.1 on bandwidth and sampling.)

1.1.9 THE FAST FOURIER TRANSFORM

A modern MRI scanner must perform a large number of DFTs to reconstruct an image. Computational speed is an important issue. Consider the DFT from Eq. (1.22)

$$D_J = \sum_{K=0}^{N-1} d_K W_N^{JK}, \qquad J = 0, 1, 2, \ldots, N-1 \qquad (1.36)$$

where we have abbreviated the twiddle factor

$$W_N = e^{\frac{-2\pi i}{N}} \tag{1.37}$$

If the twiddle factors are precalculated and stored in a table of complex numbers, then it takes approximately N^2 complex multiplications to evaluate Eq. (1.36) for all values of J. The FFT algorithm reduces the number of operations and thereby increases the computational efficiency.

If N is even, then the signal can be split into two subsignals: its even-indexed elements and its odd-indexed elements. It can be shown that the DFT can be expressed as a linear combination of two half-length DFTs of the even- and odd-indexed signals. The total number of complex multiplications is then approximately $2(N/2)^2 = N^2/2$. If N is a power of 2 $(1, 2, 4, 8, 16, 32, \ldots)$, then this process can be continued recursively all the way down to single-point DFTs. When the sub-FFTs are finally reassembled, the number of complex multiplications is on the order of $N \log_2 N$. For $N = 512 = 2^9$, this increases the speed by a factor of approximately $512/9 \approx 57$, which is very substantial. Clinical high-resolution MRI would not be feasible without the invention of the FFT.

Although 2 is by far the most common base, or *radix*, for the length of the FFT, some computational speed can be gained whenever N is not a prime number. Because of the wide availability and maximal computational efficiency of the radix-2 FFT, however, signals of arbitrary length are often extended to the next power of 2 by using zero filling (see Section 13.1.2) prior to reconstruction.

1.1.10 THE DIRAC DELTA AND NORMALIZATION OF THE FOURIER TRANSFORM

This section describes the Dirac delta function (named after Paul Adrien Maurice Dirac, 1902–1984, an English physicist and mathematician) and the normalization of the continuous FT. Those who are not interested in these mathematical properties can skip to the next section.

The Dirac delta function (Dirac 1957, 58–61) (or perhaps more properly the Dirac delta distribution) is zero everywhere except at the origin, yet has unit area:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \int_{-\infty}^{+\infty} \delta(x)\,dx = 1 & \end{cases} \tag{1.38}$$

Thus, the $\delta(x)$ must not be finite at the origin. The Dirac delta has the property of picking out a single value of a function under multiplication and integration:

$$\int_{-\infty}^{+\infty} \delta(x - x') f(x)\, dx = f(x') \tag{1.39}$$

The normalization of continuous FTs, for example, Eqs. (1.1), (1.3), and (1.4), requires that:

$$g(x) = \int_{-\infty}^{+\infty} e^{2\pi i k x} \left[\int_{-\infty}^{+\infty} e^{-2\pi i k x'} g(x')\, dx' \right] dk \tag{1.40}$$

By rearranging Eq. (1.40):

$$g(x) = \int_{-\infty}^{+\infty} g(x')\, dx' \left[\int_{-\infty}^{+\infty} e^{2\pi i k (x - x')} dk \right] \tag{1.41}$$

and comparing this with Eq. (1.39), we conclude that the contents of the square brackets in Eq. (1.41) must be a representation of the Dirac delta:

$$\delta(x - x') = \int_{-\infty}^{+\infty} e^{2\pi i k (x - x')} dk \tag{1.42}$$

or, substituting $(x - x') = a$, and $2\pi k = u$:

$$\delta(a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iau} du \tag{1.43}$$

To show that Eq. (1.43) is true, it is useful to first state the result:

$$\int_{-\infty}^{+\infty} \frac{\sin u}{u}\, du = \pi \tag{1.44}$$

which can be demonstrated either with contour or numerical integration.

Then consider:

$$\int_{-\infty}^{+\infty} f(a)\left(\int_{-g}^{g} e^{iav}dv\right)da = \int_{-\infty}^{+\infty} f(a)\left[\frac{e^{iav}}{ia}\right]_{v=-g}^{g} da$$

$$= \int_{-\infty}^{+\infty} f(a)\frac{2\sin ag}{ag} g\,da \qquad (1.45)$$

Letting $u = ag$, Eq. (1.45) becomes:

$$2\int_{-\infty}^{+\infty} f\left(\frac{u}{g}\right)\frac{\sin u}{u}\,du \qquad (1.46)$$

Letting $g \to \infty$ and using the result from Eq. (1.44):

$$2\int_{-\infty}^{+\infty} f\left(\frac{u}{g}\right)\frac{\sin u}{u}\,du \to 2f(0)\int_{-\infty}^{+\infty}\frac{\sin u}{u}\,du = 2\pi f(0) \qquad (1.47)$$

Combining Eqs. (1.45) and (1.47):

$$\int_{-\infty}^{+\infty} f(a)\left(\int_{-\infty}^{+\infty} e^{iau}du\right)da = 2\pi f(0) \qquad (1.48)$$

Thus, the integral of the complex exponential is a Dirac delta function. Specifically, Eq. (1.43) is verified, and the continuous FT and IFT pair of Eqs. (1.1) and (1.3) is properly normalized.

Selected References

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Related Sections

Section 11.1 Bandwidth and Sampling
Section 13.1 Fourier Reconstruction
Section 13.4 Partial Fourier Reconstruction

1.2 Rotating Reference Frame

The description of many physical quantities and processes requires a coordinate system, or *reference frame*. Depending on the choice of reference frame, the description of a physical process can be drastically different. For example, suppose a bicyclist is traveling east as viewed from a building's window. The same bicyclist will appear to be going west when viewed from a car traveling east at a higher speed. In this example, the building and the car are two different reference frames. They give entirely different descriptions of the same physical process.

Two reference frames, the *laboratory reference frame* and the *rotating reference frame*, are often employed to describe MRI phenomena. The laboratory reference frame is defined with respect to the scanner or the magnet. By convention, when discussing the rotating frame, the B_0-field direction is always chosen to be the z axis (also known as the *longitudinal axis*). (This is somewhat different from the definition of logical gradient axes in Section 7.3, in which z axis may or may not correspond to the B_0-field direction.) The x and y axes in the laboratory reference frame are selected as a pair of orthogonal vectors in a plane normal to the B_0-field (denoted by x' and y' in Figure 1.3a). In a horizontal magnet with a cylindrical bore, the y axis is usually chosen to be from floor to ceiling (or from down to up). When the z axis is pointing toward the viewer, the x axis is selected to be from left to right. These three axes conform to the right-hand rule, which states that if the fingers of the right hand are curled from the positive x axis to the positive y axis, the thumb points along the positive z axis. Such a coordinate system is called a *right-handed Cartesian coordinate system*, and the plane defined by the x and y axes is known as the *transverse plane*.

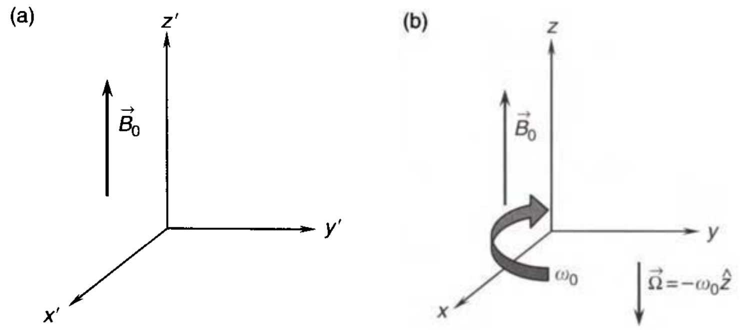

FIGURE 1.3 (a) The laboratory reference frame and (b) a rotating reference frame. These frames are related by a rotation about the z axis with an angular frequency ω_0.